A finite element–wavelet hybrid algorithm for atmospheric tomography

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Florence, Italy

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Outline

- Wavelets
- Finite element–wavelet algorithm
- Results: MCAO
- Results: LTAO
Atmospheric tomography

- Multi Conjugate AO (MCAO)
- Laser Tomography AO (LTAO)
- Multi Object AO (MOAO)
  - use several guidestars
  - goal: quality in the field of view

Atmospheric tomography: WFS measurements $\rightarrow$ layers
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\[ A\phi = s \]

Atmospheric tomography operator → turbulence layers → noisy measurements

Atmospheric tomography: WFS measurements → layers
Parametrization of layers with wavelets

**Concept:** use **wavelets** to represent **turbulence layers**

**Wavelets:**
- a way to represent and analyze signals
- used in JPEG compression
Parametrization of layers with wavelets

**Concept:** use wavelets to represent turbulence layers

**Wavelets:**
- a way to represent and analyze signals
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**Wavelets decomposition of a layer:**

- turb. layer
  - 16,384 coeff
- scales 0,1
  - 16 coeff
- scales 0,1,2
  - 64 coeff
- scales 0,…,3
  - 256 coeff
- scales 0,…,4
  - 1024 coeff
Wavelet properties

Advantages of wavelets:

- multiscalar structure
  - good approximative properties

\[ C_\varphi \approx c_{W}^{-1}D_{W}(M_{F}^{j})(\xi) = |\xi|^{-11/3}f(\xi) \]

\( D = \text{diag}(\ldots, 2^{-11/3}j, \ldots) \)

\( \text{two Daubechies 3 wavelets} \)
Wavelet properties

Advantages of wavelets:

- multiscalar structure
  - good approximative properties
- compact support
  - discrete wavelet transform (DWT)
    - DWT is $O(n)$, parallelizable!

Kolmogorov power law:

$$C_{\phi} = c F^{-1} M_{F} \approx c W^{-1} D_{W}(Mf)(\xi) = |\xi|^{-11/3} f(\xi)$$

$D_{j} = \text{diag}(..., 2^{-11/3} j, ...)$

$two Daubechies 3 wavelets$
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Kolmogorov power law:

$$C_\phi = c\mathcal{F}^{-1}MF$$

$$\left(Mf\right)(\xi) = |\xi|^{-11/3}f(\xi)$$
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Kolmogorov power law:

\[
C_\phi = c\mathcal{F}^{-1} M \mathcal{F} \\
(Mf)(\xi) = |\xi|^{-11/3} f(\xi)
\]

\[
C_\phi \simeq c W^{-1} DW \\
D = \text{diag}(\ldots, 2^{-11j}, \ldots) \\
j \ldots \text{wav scale}
\]
Minimum variance turbulence profile estimate:

\[
\begin{align*}
\text{inverse noise covariance} & \quad \text{inverse turbulence covariance} \\
\quad \downarrow & \quad \downarrow \\
(A^* C^{-1}_\eta A + C^{-1}_\phi) \phi &= A^* C^{-1}_\eta s \\
\text{atmospheric tomography} & \quad \text{turbulence layers} \quad \text{noisy measurements}
\end{align*}
\]

Alternatives:
the 3-step-approach\(^1\) (e.g., Kaczmarz iteration, Gradient method, cg–method)

\(^1\)Ramlau, Rosensteiner, Saxenhuber, Obereder
Solution methods to \((A^* C_\eta^{-1} A + C_\phi^{-1}) \phi = A^* C_\eta^{-1} s:\)

<table>
<thead>
<tr>
<th>(P)CG Based</th>
<th>MVM</th>
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<th>precond.</th>
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- PCG = preconditioned conjugate gradient

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1Gilles, Vogel, Ellerbroek
2Yang, Vogel, Ellerbroek
3Tallon, Béchet, Thiébaut et al.
Minimum variance turbulence profile estimation methods

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\(\text{MG-PCG}^1\) Gilles, Vogel, Ellerbroek
\(\text{FG-PCG}^2\) Yang, Vogel, Ellerbroek
\(\text{FrIM–3D}^3\) Tallon, Béchet, Thiébaut et al.

○ PCG = preconditioned conjugate gradient
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Solve:

\[(W A^* C_{\eta}^{-1} A W^{-1} + \alpha D^{-1}) c = W A^* C_{\eta}^{-1} s\]

- discrete wavelet transform \(O(n)\)
- atmospheric tomography operator coefficients
- bilinear basis (sparse)
- diagonal wavelet basis

Method handles:

- cone effect
- tip/tilt indetermination
- spot elongation
Solve:

\[
(W A^* C_\eta^{-1} A W^{-1} + \alpha D^{-1}) c = W A^* C_\eta^{-1} s
\]

using a conjugate gradient (CG) based method.
Solve:

\[
(M \equiv (W A^* C_\eta^{-1} A W^{-1} + \alpha D^{-1}) c = W A^* C_\eta^{-1} s)
\]

using a conjugate gradient (CG) based method.

\[
\text{cost of the iterative method} = \text{cost of applying } M \cdot \# \text{ of iterations}
\]
Finite Element–Wavelet algorithm

Solve:

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cost of applying \( M \)

- \( M \) is matrix–free
  - if applied sequentially!
- parallelizable w.r.t. layers, WFS
Finite Element–Wavelet algorithm

Solve:

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\end{center}

cost of applying \( M \)
- \( M \) is matrix–free
  if applied sequentially!
- parallelizable w.r.t. layers, WFS

\# of iterations
- warm restart
- preconditioner: Jacobi = \( \text{diag}(M) \)
- cascadic multiscale
Simulations in OCTOPUS: MCAO

Configuration:
- Telescope aperture diameter: 42 m
- 6 laser guide stars (LGS)
  - 84×84 subapertures
- 3 natural guide stars (NGS)
  - 1 sensor with 2×2 subapertures
  - 2 sensors with 1×1 subapertures
- 3 DMs
  - at 0, 4000, 12,700 m
  - 9,296 active actuators

Simulated data:
- OCTOPUS – official simulation tool of ESO
- 9 atmospheric layers
- quality evaluated in 25 directions
Quality results: MCAO

LGS flux: 50-500 photons/subap/frame, elongated spots
NGS flux: 500 photons/subap/frame
Method: 20 CG iterations

On-axis

Field average
Quality results: MCAO, with preconditioning

LGS flux: 100 photons/subap/frame, elongated spots
NGS flux: 500 photons/subap/frame
Method: 1-10 PCG iterations, 3-layer
Speed results: MCAO

System configuration:
- Intel(R) Xeon(R) CPU X5650 @ 2.67GHz
- 12 Cores (dual hexacore)

MVM
- 92 ms

Finite Element–Wavelet
- 3-layer, PCG 4 iter
- 3.0 ms

7 cores used: 6 WFS + 1 core for TTS computation
Simulations in OCTOPUS: LTAO

Configuration:

- Telescope aperture diameter: 42 m
- 6 laser guide stars (LGS)
  - 84×84 subapertures
  - in circle with 7.5 arcmin diam.
- 3 natural guide stars (NGS)
  - 84×84 subapertures
  - in circle with 10 arcmin diam.
- 1 DM
  - 5,402 active actuators

Simulated data:

- OCTOPUS – official simulation tool of ESO
- 9 atmospheric layers
- quality evaluated at the zenith
Quality results: LTAO

LGS flux: 20-500 photons/subap/frame, elongated spots
NGS flux: 300 photons/subap/frame
Method: 30-40 CG iterations
Speed results: LTAO/MOAO

System configuration:
- Intel(R) Xeon(R) CPU X5650 @ 2.67GHz
- 12 Cores (dual hexacore)

![Graph of MVM performance](image1)

- **MVM**
  - 80 ms
  - 9 cores used: 9 WFS / 9 layers

![Graph of Finite Element–Wavelet performance](image2)

- **Finite Element–Wavelet**
  - 9-layer, PCG 4 iter
  - 5.6 ms
Summary and Outlook

**Finite Element–Wavelet method**

- CG-based
- globally $O(n)$, parallelizable
  - DWT $O(n)$, parallelizable
  - efficient representation of turbulence statistics
  - cascadic multiscale

**Outlook**

- application to MOAO (open loop)
- combining preconditioning, multiscale and other ideas → reduce # of iter
- improving parallelization

Thanks for your attention!
M. Yudytskiy (RICAM)
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